ON THE MAPPING CLASS GROUP OF SIMPLE 3-MANIFOLDS

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Let M denote an orientable and compact 3-manifold which is irreducible, boundary-irreducible and sufficiently large (notations as in [Wa 3]). Then M wi be called *simple* if (in the notations of [Wa 3]) every incompressible annulus or torus in M is boundary-parallel.

The object of this paper is to prove (see 3.2.):

Theorem. *If M is a simple 3-manifold, then the mapping class group of M is finite.*

It is known that every homotopy equivalence $f : M_1 \to M_2$ between simple 3-manifolds can be deformed into a homeomorphism (see [Jo 1], [Jo 2], or [Swa 1]). Moreover, every isomorphism $\phi : \pi_1 M_1 \to \pi_1 M_2$ is induced by a homotopy equivalence. Hence we have the following

Corollary. *If M is a simple 3-manifold, then the outer automorphism group of $\pi_1 M$ is a finite group.*

To give a concrete example, let K be any non-trivial knot in S^3 without companions (in the sense of [Schu 1]). Then the outer automorphism group of the knot group of K is a finite group, and the knot space of K admits only finitely many homeomorphisms, up to isotopy.

In order to prove the theorem, we shall use the concept of characteristic submanifolds as developed in [Jo 2], together with certain finiteness theorems of Haken and Hemion.

§ 1 Notations and preliminaries

Throughout this paper we work in the PL-category, and entirely in the framework of "manifolds with boundary-patterns" and "admissible maps" as used in [Jo 2]. For convenience we here repeat the necessary definitions.

Let M be a compact n-manifold, $n \leq 3$. A *boundary-pattern* for M consists of a set $\underline{\underline{m}}$ of compact, connected $(n-1)$-manifolds in ∂M, such that the intersection of any i, $i = 2,3,4$, of them consists of $(n-i)$-manifolds.

The elements of $\underline{\underline{m}}$ are called the *faces* of $(M,\underline{\underline{m}})$, and the *completed boundary-pattern* of $(M,\underline{\underline{m}})$ is defined to be the set $\underline{\underline{m}} \cup \{$components of $(\partial M - \bigcup_{G\in\underline{\underline{m}}} G)^{-} \}$. A boundary-pattern is *complete* if it is equal to its completed boundary-pattern.

An *admissible map* $f : (N,\underline{\underline{n}}) \to (M,\underline{\underline{m}})$ is a map $f : N \to M$ satisfying

$$\underline{\underline{n}} = \dot{\bigcup}_{G\in\underline{\underline{m}}} \text{ components of } f^{-1}G \qquad (\dot{\bigcup} = \text{disjoint union}).$$

An *admissible homotopy* is a continuous family of admissible maps. Having defined "admissible homotopy" one also has defined "admissible isotopy".

An *i-faced disc,* $i \geq 1$, denotes a 2-disc with complete boundary-pattern and i faces.

A boundary-pattern $\underline{\underline{m}}$ of M is *useful* if every admissible map $f : (D,\underline{\underline{d}}) \to (M,\underline{\underline{m}})$ can be admissibly deformed near a point, where $(D, \underline{\underline{d}})$ is an i-faced disc, $1 \leq i \leq 3$. It is a theorem [Jo 2] that this is equivalent to the statement that the boundary of every admissibly embedded i-faced disc, $1 \leq i \leq 3$, bounds a disc in ∂M such that $D \cap \bigcup_{G\in\underline{\underline{m}}} \partial G$ is the cone on $\partial D \cap \bigcup_{G\in\underline{\underline{m}}} \partial G$.

An admissible curve $f : (k, \partial k) \to (M,\underline{\underline{m}})$, where $k = I$ or S^1, is called *essential,* if it cannot be admissibly deformed near a point. An admissible map $f : (N,\underline{\underline{n}}) \to (M,\underline{\underline{m}})$ is called *essential,* if it maps essential curves to essential curves.

A 3-manifold will mean an orientable 3-manifold (not necessarily connected) 2-manifolds (surfaces) are not generally required to be orientable or connected Whenever the notation of an *annulus* or *Möbius band* appears, it is to be understood that the boundary-pattern is the collection of its boundary-curves. A *square* is a 4-faced disc. An *inner square* or *annulus* in a surface $(F,\underline{\underline{f}})$ is an essential surface $(A,\underline{\underline{a}})$ in $(F,\underline{\underline{f}})$ such that $(A,\underline{\underline{a}})$, together with its completed boundary-pattern, is a square or annulus.

A 3-manifold $(M,\underline{\underline{m}})$ will be called *I-bundle* or *Seifert fibre space* if M admits such a structure (see [Sei 1] [Wa 1]), with fibre projection $p : M \to F$, in such a way that the sides of $(M,\underline{\underline{m}})$ are either components of $(\partial M - p^{-1}\partial F)^{-}$ or consist entirely of fibres.

The following submanifolds will play a crucial role throughout this paper.

1.1. *Definition. Let* $(M,\underline{\underline{m}})$ *be a 3-manifold. An* essential F-manifold, *W in* $(M,\underline{\underline{m}})$ *is an embedded 3-manifold such that*

1. m *induces a boundary-pattern of* W *such that every component of* W *is either an I-bundle or a Seifert fibre space.*
2. $(\partial W - \partial M)^-$ *is essential in* $(M,\underline{\underline{m}})$.

1.2. *Definition. An essential* F*-manifold,* V*, in* $(M,\underline{\underline{m}})$ *is called a* characteristic submanifold *if the following holds:*

1. *if* W *is any essential* F*-manifold in* $(M,\underline{\underline{m}})$*, then* W *can be admissibly isotoped into* V*, and*
2. *if* W' *is any union of components of* $(M-V)^-$*, then* $V \cup W'$ *is not an essential* F*-manifold.*

The following facts about characteristic submanifolds will be used in this paper without proof. They are shown in [Jo 2].

1.3. *Theorem. Let* $(M,\underline{\underline{m}})$ *be an irreducible, sufficiently large 3-manifold with useful and complete boundary-pattern. Then the characteristic submanifold in* $(M,\underline{\underline{m}})$ *exists and is unique, up to admissible ambient isotopy.*

1.4. *Theorem. Let* $(M,\underline{\underline{m}})$ *be given as in* 1.3. *and let* V *be the characteristic submanifold in* $(M,\underline{\underline{m}})$*. Let* $M' = (M-V)^-$ *and let* $\underline{\underline{m}}' = \{G \mid G$ *is component either of* $M' \cap V$*, or of* $G \cap H$*, where* $H \in m\}$.

Then for every essential square, annulus, or torus, T*, in* $(M',\underline{\underline{m}}')$ *either 1. or 2. holds*

1. $T \cap V \neq \emptyset$ *and the component of* $(M',\underline{\underline{m}}')$ *which contains* T *is admissibly homeomorphic to* $T \times I$.
2. $T \cap W = \emptyset$. *and* T *is admissibly parallel in* $(M',\underline{\underline{m}}')$ *to a side of* $(M',\underline{\underline{m}}')$ *which is contained in* $(\partial V - \partial M)^-$.

§ 2 *Homeomorphisms of I-bundles*

In order to prove the theorem given in the introduction, we need a certain technical result on homeomorphisms of I-bundles. This will be established in 2.3.

Let G_1 and G_2 be two essential surfaces in $(F,\underline{\underline{f}})$ such that $(\partial G_1 - \partial F)^-$ and $(\partial G_2 - \partial F)^-$ are transversal. Then we say that G_2 is in a *very good position with respect to* G_1, provided the number of points of $(\partial G_2 - \partial F)^- \cap (\partial G_2 - \partial F)^-$ cannot be diminished and the number of components of $(\partial G_2 - \partial F)^-$ contained in G_1 cannot be enlarged, using an admissible isotopy of G_2.

Now, for the following, let $(X,\underline{\underline{x}})$ denote an I-bundle (twisted or not) with complete boundary-pattern, and with projection $p : X \to B$. Let F be the union of the lids of $(X,\underline{\underline{x}})$, i.e. $F = (\partial X - p^{-1}\partial B)^-$, and let $\underline{\underline{f}}$ be the boundary-pattern of F induced by $\underline{\underline{x}}$. Finally, denote by $d : (F,\underline{\underline{f}}) \to (F,\underline{\underline{f}})$ the admissible involution given by the reflections in the I-fibres in X.

2.1. *Lemma.* *Let* G *be an essential surface in* $(F,\underline{\underline{f}})$. *Suppose that* G *is in a very good position with respect to* dG. *Let* $h : (X,\underline{\underline{x}}) \to (X,\underline{\underline{x}})$ *be an admissible homeomorphism with* $h|(F-G)^- = \mathrm{id}|(F-G)^-$. *Then there is an admissible isotopy* h_t, $t \in I$, *of* $h = h_0$, *with* $h_t(G) = G$, *for all* $t \in I$, *such that*

$$h_1|(\partial d G - \partial F)^- = \mathrm{id}|(\partial d G - \partial F)^- \text{ and } h_1|(F-G)^- = \mathrm{id}|(F-G)^- .$$

Proof. Denote by $k_1,\ldots,k_n$, $n \geq 1$, all the components of $(\partial G - \partial F)^-$. Suppose that $h \mid d k_1 \cup \ldots \cup d k_j = \mathrm{id}$, for some $j \geq 1$, and consider $k = k_{j+1}$. It remains to show the existence of an admissible isotopy h_t, $t \in I$, of $h = h_0$, with $h_t(G) = G$, for all $t \in I$, such that

$$h_1 | d k_1 \cup \ldots \cup d k_j \cup d k = \mathrm{id} \text{ and } h_1|(F-G)^- = \mathrm{id}|(F-G)^- .$$

k is an essential curve (closed or not) in $(F,\underline{\underline{f}})$ since G is an essential surface in $(F,\underline{\underline{f}})$. The preimage $p^{-1}(p\,k)$ is, in general, not a square or annulus, for $k \cap dk$ need not be empty. But it is easy to see that there is always an I-fibre preserving immersion $g_k : k \times I \to X$ with $g_k(k \times I) = p^{-1}p\,k$, and $g_k(k \times 0) = k$ and $g_k(k \times 1) = dk$.

Define $l = h^{-1}(dk)$. Observe that $h^{-1}|k = id|k$, for $h|(F-G)^- = id|(F-G)^-$, and that the immersion g_k and the *homeomorphism* h are both essential maps. Hence $h^{-1}g_k$ is an essential singular square or annulus in $(X,\underline{\underline{x}})$ with k as one side. This implies that $h^{-1}g_k$ can be admissibly deformed (rel $k \times 0$) in $(X,\underline{\underline{x}})$ into a vertical map, i.e. into g_k. To see this observe that $p \cdot h^{-1}g_k$ can be admissibly contracted (rel $k \times 0$) in the base B into $ph^{-1}g_k$ $(k \times 0)$, and lift such a contraction to an admissible homotopy of $h^{-1}g_k$. The restriction of this homotopy to $(k \times 1) \times I$ defines an admissible deformation $f : k \times I \to F$ with $f|k \times 0 = id|l$ and $f|k \times 1 = id|dk$.

Case 1 $dk \cap (\partial G - \partial F)^-$ *is empty.*

In this case dk does not meet $\cup_{1\leq i\leq j} dk_i$ or $(\partial G - \partial F)^-$. The same holds for l : that l does not meet $\cup_{1\leq i\leq j} dk_i$ follows from $h|\cup_{1\leq i\leq j} dk_i = id$, $h(l) = dk$, and $\cup_{1\leq i\leq j} dk_i = \emptyset$, and that l does not meet $(\partial G - \partial F)^-$ follows from $h|(F-G)^- = id$. Hence, since G is in a very good position with respect to dG, it follows that f can be admissibly deformed (rel $k \times \partial I$) so that afterwards $S = f^{-1}((\partial G - \partial F)^- \cup \cup_{1\leq i\leq j} dk_i)$ is a system of pairwise disjoint curves which are admissibly parallel to the side $k \times 0$ of $k \times I$ (transversality lemma; see [Wa 2]).

If S is empty and dk lies in $(F-G)^-$, there is nothing to show since $h|(F-G)^- = id|(F-G)^-$. If S is empty and dk lies in G, the existence of the required isotopy h_t, $t \in I$, follows from Baer's theorem (see §1 of [Wa 3]).

Thus we may suppose that S is non-empty. Then S splits $k \times I$ into a non-trivial system of squares or annuli. Let A' be that one of them which contains $k \times 1$. As usual, using the theorem of Nielsen (see §1 of [Wa 3]), the existence of the map $f|A'$ implies the existence of an inner square or annulus A in $(F,\underline{\underline{f}})$ with $(\partial A - \partial F)^- = t \cup dk$, where t is either dk_i, for some $1 \leq i \leq j$, or a component of $(\partial G - \partial F)^-$. Moreover, it follows from our choice of A' that $A^\circ \cap ((\partial G - \partial F)^- \cup \cup_{1\leq i\leq j} dk_i) = \emptyset$. Consider $h^{-1}A$, and note that $h^{-1}t = t$, for $h|\cup_{1\leq i\leq j} dk_i = id$ and $h|(F-G)^- = id$. Hence $h^{-1}A$ is also an inner square or annulus in $(F,\underline{\underline{f}})$ with $(h^{-1}A)^\circ \cap ((\partial G - \partial F)^- \cup \cup_{1\leq i\leq j} dk_i) = \emptyset$, and $(\partial h^{-1}A - \partial F)^- = h^{-1} dk \cup h^{-1}t = l \cup t$. Now l is admissibly isotopic, via $h^{-1}A$, to t and then, via A, to dk. Extending these isotopies in the obvious way, we get the required isotopy h_t, $t \in I$, provided h does not interchange the components of $(\partial U(t) - \partial F)^-$, where $U(t)$ is some regular neighbourhood of t with $h(U(t)) = U(t)$.

But the latter must be true, for otherwise h reverses the orientation of F which would imply that $G = F$ since $h|(F-G)^- = id|(F-G)^-$.

Case 2 $dk \cap (\partial G - \partial F)^-$ *is non-empty*

G is an essential surface in $(F,\underline{\underline{f}})$ which is in a very good position with respect to dG. Hence we may suppose that f is admissibly deformed (rel $k \times \partial I$) so that $f^{-1}(\partial G - \partial F)^-$ is a system of curves which join $k \times 0$ with $k \times 1$.

We first consider the subcase that $f^{-1}(\cup_{1\le i\le j} dk_i)$ is empty. Let a_1 be a component of $(1 - G)^-$, and let F_1 be the component of $(F-G)^-$ which contains a_1. Then a_1 is an essential arc in F_1, for G is in a very good position with respect to dG. $f|a_1 \times I$ is an admissible homotopy of a_1 in F_1, and since $h|(F-G)^- = id|(F-G)^-$ we have $a_1 = f(a_1 \times 0) = f(a_1 \times 1)$. If $f|a_1 \times 1$ cannot be admissibly deformed (rel $a_1 \times \partial I$) into a_1, then, by Nielsen's theorem, F_1 has to be an inner annulus in $(F,\underline{\underline{f}})$. Moreover, it follows that $F_1 \cap \cup_{1\le i\le j} dk_i = \emptyset$ since $f^{-1}(\cup_{1\le i\le j} dk_i) = \emptyset$. Sliding a_1 around F_1 (if necessary), we may suppose that h is isotoped so that $f|a_1 \times I$ now has the preceding property, for all components a_1 of $(dk - G)^-$ In this situation, the existence of the homotopy f shows that every component b_2 of $1 \cap G$ can be admissibly deformed in G into a component a_2 of $k \cap G$, using a deformation which is constant on ∂b_2 and which does not meet $\cup_{1\le i\le j} dk_i$. In fact, by Baer's theorem, these deformations may be chosen as isotopies. Extending all these isotopies in the obvious way, we get the required isotopy h_t, $t \in I$.

Now let us suppose that f cannot be admissibly deformed so that afterwards $f^{-1}(\cup_{1\le i\le j} dk_i) = \emptyset$ and that $f^{-1}(\partial G - \partial F)^-$ consists of curves which join $k \times 0$ with $k \times 1$. Then f can be admissibly deformed (rel $k \times \partial I$) so that $f^{-1}(\cup_{1\le i\le j} dk_i)$ is a non-empty system of curves which are parallel to $k \times 0$. This system splits $k \times I$ into squares or annuli. Let A' be that one of them which contains $k \times 1$. Using this A' the existence of the required isotopy h_t follows by a similar argument as in Case 1. q.e.d.

For the next lemma let G be again an essential surface in $(F,\underline{\underline{f}})$, and suppose that G is in a very good position with respect to dG. Let U be a regular neighbourhood of $(\partial G - \partial F)^-$ in $(F,\underline{\underline{f}})$. Denote by C the *essential union* of U and dU, i.e. the smallest essential surface in $(F,\underline{\underline{f}})$ containing $U \cup dU$.

2.2 *Lemma.* *Suppose that* $(X,\underline{\underline{x}})$ *is not the I-bundle over the annulus, torus, Möbius band, or Klein bottle. Let* $h : (X,\underline{\underline{x}}) \to (X,\underline{\underline{x}})$ *be an admissible homeomorphism with* $h|(F-G)^- = id|(F-G)^-$.

Then there is an admissible isotopy h_t, $t \in I$, *of* $h = h_0$, *with* $h_t(G) = G$, *for all* $t \in I$, *such that* $h_1|p^{-1}pC = id$ *and* $h_1|(F-G)^- = id$.

Proof. By 2.1., we may suppose that $h|(\partial G - \partial F)^- = id$ and $h|(\partial dG - \partial F)^- = id$. $h|F$ is orientation preserving, for $h|(F-G)^- = id$. Hence we may suppose that $h|U = id$ and $h|d\,U = id$, and hence also $h|C = id|C$. Observe that, by our choice of C, $dC = C$.

Denote $N = p^{-1}pC$, and let $\underline{\underline{n}}$ be the boundary-pattern of N induced by $\underline{\underline{x}}$. Then the fibration of $(X,\underline{\underline{x}})$ induces an admissible fibration of $(N,\underline{\underline{n}})$ as a system of I-bundles. C is then the union of all the lids of these I-bundles.

By its very definition, C is an essential surface in $(F,\underline{\underline{f}})$. Hence it follows that each component A of $(\partial N - \partial X)^-$ is an essential square or annulus in $(X,\underline{\underline{x}})$. Since $h|F \cap \partial A = id|F \cap \partial A$, we have that $h|A$, together with $id|A$, defines an admissible singular annulus or torus in $(X,\underline{\underline{x}})$. Applying Nielsen's theorem to the product of this map with p, we find that this singular annulus or torus has to be inessential in $(X,\underline{\underline{x}})$ (recall our suppositions on $(X,\underline{\underline{x}})$). This in turn implies that $h|A$ can be admissibly deformed (rel $F \cap \partial A$) in $(X,\underline{\underline{x}})$ into A (we are in an I-bundle). By 5.5 of [Wa 3], this deformation may be chosen as an isotopy, and this isotopy can be extended to an admissible isotopy of h, which is constant on F. Therefore it follows that h can be admissibly isotoped (rel F) so that afterwards, $h(N) = N$.

Let $(N_1,\underline{\underline{n}}_1)$ be any component of $(N,\underline{\underline{n}})$, and let $\bar{\underline{\underline{n}}}_1$ be the completed boundary-pattern of $(N_1\underline{\underline{n}}_1)$. Then $h|N_1 : (N_1,\bar{\underline{\underline{n}}}_1) \to (N_1,\bar{\underline{\underline{n}}}_1)$ is an admissible homeomorphism with $h|F \cap N_1 = id|F \cap N_1$. If $(N_1,\bar{\underline{\underline{n}}}_1)$ is not the I-bundle over the annulus or Möbius band, it follows, by an argument of 3.5. of [Wa 3], that $h|N_1 : (N_1,\bar{\underline{\underline{n}}}_1) \to (N_1,\bar{\underline{\underline{n}}}_1)$ can be admissibly isotoped into the identity, using an isotopy which is constant on $N_1 \cap F$. This is also true if $(N_1,\bar{\underline{\underline{n}}}_1)$ is the I-bundle over the Möbius band. To see this, note that in this case N_1 is a regular neighbourhood of a vertical Mobius band. Moreover, every homeomorphism of the Mobius band which is the identity on the boundary is isotopic (rel boundary) to the identity.

Let $\overline{N}$ be a union of components of N such that $h|\overline{N} = id|\overline{N}$. By what we have seen so far, we may suppose that $\overline{N}$ is chosen so that $N - \overline{N}$ consists of I-bundles over the annulus.

So let N_1 be any component of $N - \overline{N}$. Then $(N_1, \bar{\underline{\underline{n}}}_1)$ is an I-bundle over the annulus and we may suppose that $h|N_1 : (N_1,\bar{\underline{\underline{n}}}_1) \to (N_1,\bar{\underline{\underline{n}}}_1)$ cannot be admissibly isotoped to the identity, using an isotopy which is constant on $N_1 \cap F$. It remains to show that there is an admissible isotopy h_t, $t \in I$, of $h = h_0$, with $h_t(G) = G$ and $h_t(N) = N$, such that $h_1|\overline{N} \cup N_1 = id|\overline{N} \cup N_1$ and $h_1|(F-G)^- = id|(F-G)^-$.

For this consider N_1 as a regular neighbourhood of a vertical annulus A_1 in $(X,\underline{\underline{x}})$. Without loss of generality, one boundary component of A_1, say k_1, is a component of $(\partial G - \partial F)^-$ and the other one, say k_2, is contained either in G or in $(F-G)^-$ without meeting $(\partial G - \partial F)^-$ (recall our choice of N_1).

If k_2 lies in G, observe that $h|A_1 : A_1 \to A_1$ is isotopic to the identity, using an isotopy which is constant on k_1. Extending such an isotopy to an admissible isotopy of h which is constant outside a regular neighbourhood of N_1, we find the required isotopy h_t.

If k_2 lies in $(F-G)^-$, then ∂A lies in $(F-G)^-$. It follows that, for one component X_1 of $(X - N)^-$ which meets N_1, all lids are contained in $(F-G)^-$. Let B be an essential vertical square in $(X_1,\bar{\underline{\underline{x}}}_1)$ which meets N_1, where $\underline{\underline{x}}_1$ is the boundary-pattern induced by $\underline{\underline{x}}$ and $\bar{\underline{\underline{x}}}_1$ the completed boundary-pattern of $(X_1,\underline{\underline{x}}_1)$. Since $h|(F-G)^- = id|(F-G)^-$, we have that $h|B$, together with $id|B$, defines an admissible singular annulus in $(X_1,\bar{\underline{\underline{x}}}_1)$. By our suppositions on $h|N_1$, this singular annulus is essential in $(X_1,\bar{\underline{\underline{x}}}_1)$ and cannot be admissibly deformed into a vertical map. Hence, by Nielsen's theorem, $(X_1,\bar{\underline{\underline{x}}}_1)$ is the I-bundle over the annulus or Möbius band. But it cannot be the I-bundle over the Möbius band, for $h|X_1 : (X_1,\bar{\underline{\underline{x}}}_1) \to (X_1,\bar{\underline{\underline{x}}}_1)$ cannot be admissibly isotoped (rel F) into the identity since, by supposition, $h|N_1$ cannot.

By what we have seen so far, $(X_1,\bar{\underline{\underline{x}}}_1)$ has to be the I-bundle over the annulus. Moreover, $h|X_1 : (X_1,\bar{\underline{\underline{x}}}_1) \to (X_1,\bar{\underline{\underline{x}}}_1)$ cannot be admissibly isotoped into the identity, using an isotopy which is constant on $X_1 \cap F$. Thus, in particular, X_1 cannot meet $\overline{N}$. So, either $(\partial X_1 - \partial X)^-$ is connected or X_1 meets a component N_2 of N which is also an I-bundle over the annulus. Again, consider N_2 as a regular neighbourhood of a vertical annulus A_2 in $(X,\underline{\underline{x}})$. Without loss of generality, one boundary component, say l_1, of A_2

is a component of $(\partial G - \partial F)^-$ and so the other one, say l_2, is a component of $(\partial d\,G - \partial F)^-$. Since X_1 is an I-bundle over the annulus, it follows that k_1 and l_1, resp. k_1 and l_2, bound an inner annulus in $(F,\underline{\underline{f}})$. Since G is in a very good position to dG, it follows that k_1 and l_1 bound an inner annulus, i.e. k_1 lies in a component G_1 of $(F-G)^-$ which is an inner annulus in $(F,\underline{\underline{f}})$. Let H be the lid of $N_1 \cup X_1 \cup N_2$ which contains k_2. Observe that $h|N_1 \cup X_1 \cup N_2$ is admissibly isotopic in $N_1 \cup X_1 \cup N_2$ to the identity, using an isotopy which is constant on H. Extending this isotopy to an admissible isotopy of h which is constant outside of a regular neighbourhood of $N_1 \cup X_1 \cup N_2$ we find the required isotopy h_t. q.e.d.

Again let G be an essential surface in $(F,\underline{\underline{f}})$, and suppose that G is in a very good position with respect to dG.

2.3. *Proposition. Suppose that* $(X,\underline{\underline{x}})$ *is not the I-bundle over the annulus, Mobius band, torus, or Klein bottle. Let* $h : (X,\underline{\underline{x}}) \to (X,\underline{\underline{x}})$ *be an admissible homeomorphism with* $h|(F-G)^- = \mathrm{id}$.

Then there is an admissible isotopy h_t, $t \in I$, *of* $h = h_0$, *with* $h_t(G) = G$, *for all* $t \in I$, *such that* $h_1|p^{-1}p\,H = \mathrm{id}|p^{-1}p\,H$, *where* H *is the essential union of* $(F-G)^-$ *and* $(F - dG)^-$.

Proof. Let U be the regular neighbourhood of $(\partial G - \partial F)^-$ in $(F,\underline{\underline{f}})$, and define C to be the essential union of U and dU. Then, by 2.2., we may suppose that $h|p^{-1}pC = \mathrm{id}|p^{-1}pC$ and $h|(F-G)^- = \mathrm{id}|(F-G)^-$. Let H' be the union of C with all the components of $(F-C)^-$ which lie either in $(F-G)^-$ or in $(F - dG)^-$. Then observe that H is contained in H'.

Let X_1 be any component of $(X - p^{-1}p\,C)^-$ with $X_1 \cap F \subset H'$. Then, by the definition of H', at least one lid of X_1 lies in $(F-G)^-$. It suffices to show that $h|X_1 : X_1 \to X_1$ is admissibly isotopic to the identity, using an isotopy which is constant on $(\partial X_1 - \partial X)^-$ and all the lids of X_1 which lie in $(F-G)^-$. By our suppositions on h, this follows by an argument of 3.5 of [Wa 3] (recall our choice of X). q.e.d.

§ 3 *The proof of the theorem*

3.1. Lemma. Let $(M,\underline{\underline{m}})$ *be a twisted I-bundle,* $(N,\underline{\underline{n}})$ *be a product I-bundle, and* $p = (N,\underline{\underline{n}}) \to (M,\underline{\underline{m}})$ *be an admissible 2-sheeted covering. Then every admissible homeomorphism,* h, *of* $(M,\underline{\underline{m}})$ *can be lifted to an admissible homeomorphism,* g, *of* $(N,\underline{\underline{n}})$ *i.e.* $p \cdot h = g \cdot p$.

Proof. One first proves as in (5.5.) of [Wa 1] that every homeomorphism of $(M,\underline{\underline{m}})$ can be admissibly isotoped into a fibre preserving one. Hence 3.1. is proved if we show the statement of 3.1., for every non-orientable surface $(F,\underline{\underline{f}})$ and orientable 2-sheeted covering $q : (G,\underline{\underline{g}}) \to (F,\underline{\underline{f}})$. For this we may restrict ourselves to the case that $\underline{\underline{f}}$ consists of all the boundary curves of F Then we may identify each boundary component to a point. Hence we suppose that F is closed and that h is a homeomorphism which maps a set of points $x_1,\ldots, x_n$, $n \geq 0$, to itself. Using [Li 1] and [Li 2], it is not difficult to show (see [Jo 2] that h is isotopic (rel x_i) to some product of the following homeomorphisms:

1. f is a Y-homeomorphism (in the sense of [Li 2]) with $g(x_i) = x_i$, for all $1 \leq i \leq n$.

2. α_{ij}, $1 \leq i < j \leq n$, is the rotation along a fixed simple closed, 2-sided curve, k, in F, with $k \cap \cup x_i = x_i \cup x_j$, which interchanges x_i with x_j and which is the identity outside of a regular neighbourhood of k.

3. β_i, $1 \leq i \leq n$, is the end of an isotopy which moves the point x_i once around a fixed simple closed, 1-sided curve, k, in F, with $k \cap \cup x_i = x_i$, and which is constant outside of a regular neighbourhood of k.

4. g is a Dehn twist, i.e. a homeomorphism which is the identity outside a regular neighbourhood of a fixed simple closed, 2-sided curve, k, with $k \cap \cup x_i = \emptyset$.

Observe that the preimage under g of every 1-sided, simple closed curve is connected since G is orientable. Using this fact, it is an easy exercise to show that all homeomorphisms of 1. - 3. can be lifted. Now we claim that the preimage under q of every 2-sided, simple closed curve k, in F is disconnected. To see this, fix a regular neighbourhood U(k) of k, and let $U(p^{-1}k)$ be the preimage of U(k) under p. The non-trivial covering translation, d, maps $U(p^{-1}k)$ to itself. Moreover, it follows that the restriction $d\,U(p^{-1}k)$ is orientation-reserving since d is orientation-reversing. Hence, since d is a fixpoint free, d interchanges the boundary components of $U(p^{-1}k)$, if $U(p^{-1}k)$ is connected (i.e. an annulus). But this is impossible since k is 2-sided. Thus our claim follows, and so, of course, every Dehn twist of F can be lifted to G. q.e.d.

An irreducible and sufficiently large 3-manifold $(M,\underline{\underline{m}})$ is called *simple* if every essential square, annulus, or torus in $(M,\underline{\underline{m}})$ is admissibly parallel to some side of $(M,\underline{\underline{m}})$. *The mapping class group* $H(M,\underline{\underline{m}})$ is defined to be the group of all admissible homeomorphisms of $(M,\underline{\underline{m}})$ modulo admissible isotopy.

3.2. *Theorem. Let* $(M,\underline{\underline{m}})$ *be a simple 3-manifold with complete and useful boundary-pattern. Then* $H(M,\underline{\underline{m}})$ *is a finite group.*

Proof. The proof is based on the following two finiteness theorems:

1. in a simple 3-manifold there are, up to admissible isotopy, only finitely many essential surfaces of a given admissible homeomorphism type. This follows from [Ha 1].
2. the theorem is true for Stallings fibrations which are simple 3-manifolds. This follows from [He 1].

As a first consequence of these two facts, we show that the mapping class group of all simple Stallings manifolds is finite. Here a Stallings manifold means a 3-manifold $(M,\underline{\underline{m}})$ which contains an essential surface F such that $(M - U(F))^-$ consists of I-bundles, where $U(F)$ denotes a regular neighbourhood of F in $(M,\underline{\underline{m}})$. By 2. above, we may suppose that $(M - U(F))^-$ consists of two twisted I-bundles, say $M_1\ M_2$. M_1 and M_2 have product I-bundles $\tilde{M}_1$, $\tilde{M}_2$, respectively, as 2-sheeted coverings. Attaching the lids of $\tilde{M}_1$ and $\tilde{M}_2$ in the obvious way, we obtain a manifold $\tilde{M}$ and a 2-sheeted covering $p : \tilde{M} \to M$. By 1. above, it suffices to show that the subgroup of $H(M,\underline{\underline{m}})$ generated by all admissible homeomorphisms $h : (M,\underline{\underline{m}}) \to (M,\underline{\underline{m}})$ with $h(F) = F$ is finite. Since $\underline{\underline{m}}$ is a finite set, we may restrict ourselves to the case that $\underline{\underline{m}}$ is the set of all boundary components of M. $h|M_i$ is an admissible homeomorphism of M_i, and so, by 3.1., it can be lifted to an admissible homeomorphism $\tilde{h}_i$ of $\tilde{M}_i$. The two liftings $\tilde{h}_1$ and $\tilde{h}_2$ define a lifting $h : \tilde{M} \to \tilde{M}$. By construction $\tilde{M}$ is a Stallings fibration, and, by the annulus- and torus theorem (see [Wa 4], [CF 1], [Fe 1], [JS 1], [Jo 2]), it is a simple 3-manifold. Hence, by 2. above, there are only finitely many homeomorphisms

$\tilde{h} : \tilde{M} \to \tilde{M}$, up to isotopy. Hence it remains to prove that $\tilde{h}$ is isotopic to the identity if and only if h is. This in turn follows from (7) of [Zi 1]. Indeed, all suppositions of (7) of [Zi 1] are satisfied: a homeomorphism of $\tilde{M}$ is isotopic to the identity if and only if it is homotopic to the identity [Wa 3]. Moreover, the centralizer of $p_*\pi_1\tilde{M}$ is trivial in $\pi_1 M$. For otherwise $\pi_1\tilde{M}$ has non-trivial centre since $\pi_1 M$ is torsion-free [Wh 1] [Ep 1] and since $p_*\pi_1\tilde{M}$ has finite index in $\pi_1 M$. Then, by [Wa 2], $\tilde{M}$ has to be a Seifert fibre space, and so also M (see [Jo 2]). But this is a contradiction to the fact that M is a simple 3-manifold.

Now we come to the proof of the general case. It is by an induction on a great hierarchy. A great hierarchy is inductively defined as follows:

First denote $(M_1, \underline{\underline{m}}_1) = (M, \underline{\underline{m}})$. Then $\underline{\underline{m}}_1$ is a complete and useful boundary-pattern of M_1.

In (M_{2i+1}, m_{2i+1}), $i \geq 0$, we take the characteristic submanifold V_{2i+1} and we define $M_{2i+1} = (M_{2i+1} - V_{2i+1})^-$. $\underline{\underline{m}}_{2i+1}$ and the components of $(\partial V_{2i+1} - \partial M_{2i+1})^-$ induce a boundary-pattern $\underline{\underline{m}}_{2i+2}$ of M_{2i+2}. Then $\underline{\underline{m}}_{2i+2}$ is a complete and useful boundary-pattern if m_{2i+1} is.

In (M_{2i}, m_{2i}) $i \geq 1$, we pick some essential surface, F_{2i}, $F_{2i} \cap \partial M_{2i} = \partial F_{2i}$, which is not admissibly parallel to some side of (M_{2i}, m_{2i}). Such a surface always exists, if M_{2i} is not a ball (see [Wa 2] and [Jo 2]). Define $M_{2i+1} = (M_{2i} - U(F_{2i}))^-$. $\underline{\underline{m}}_{2i}$ and the components of $(\partial U(F_{2i}) - \partial M_{2i})^-$ induce a boundary-pattern $\underline{\underline{m}}_{2i+1}$ of M_{2i+1}. Then again $\underline{\underline{m}}_{2i+1}$ is a complete and useful boundary-pattern if $\underline{\underline{m}}_{2i}$ is.

By a result of Haken [Ha 2], there is an integer $n \geq 1$ such that $(M_n, \underline{\underline{m}}_n)$ consists of balls with complete and useful boundary-patterns.

If $j \geq 1$ is an even integer, denote by $H(M_j, \underline{\underline{m}}_j, F_j)$ the subgroup of $H(M_j, \underline{\underline{m}}_j)$ generated by all the admissible homeomorphisms of $(M_j, \underline{\underline{m}}_j)$ which preserve $U(F_j)$. Of course, $H(M_n, \underline{\underline{m}}_n)$ is a finite group, and so, by the facts quoted in the beginning of the proof, it suffices to prove the following:

3.3. Lemma. If $H(M_{2i+2}, \underline{\underline{m}}_{2i+2})$ *is finite, and if* M_{2i} *is not a Stallings manifold, then* $H(M_{2i}, \underline{\underline{m}}_{2i}, F_{2i})$ *is finite.*

To begin with we simplify the notations somewhat, and we write $(N_0, \underline{\underline{n}}_0) = (M_{2i}, \underline{\underline{m}}_{2i})$, $(N_1, \underline{\underline{n}}_1) = (M_{2i+1}, \underline{\underline{m}}_{2i+1})$, and $(N_2, \underline{\underline{n}}_2) = (M_{2i+2}, \underline{\underline{m}}_{2i+2})$.

Moreover, denote $F = F_{2i}$ and $H = (\partial U(F) - \partial N_0)^-$. $\underline{\underline{n}}_0$ together with the components of H, induces a boundary-pattern of the regular neighbourhood $U(F)$ which makes $U(F)$ into a product I-bundle.

By 1.3., the characteristic submanifold of a 3-manifold is unique, up to admissible ambient isotopy. This means that every admissible homeomorphism of $(N_1, \underline{\underline{n}}_1)$ can be admissibly isotoped so that it preserves the characteristic submanifold V_1 of $(N_1, \underline{\underline{n}}_1)$. This, together with the suppositions of Lemma 3.3., implies the following: there are finitely many admissible homeomorphisms $g_1, \ldots, g_m$ of $(N_0, \underline{\underline{n}}_0)$ with $g_j(U(F)) = U(F)$, for all $1 \le j \le m$, such that for a given admissible homeomorphism g, $g \in H(N_0, \underline{\underline{n}}_0, F)$, $g|N_1$ can be admissibly isotoped in $(N_1, \underline{\underline{n}}_1)$ so that afterwards

$$g|(N_1 - V_1)^- = g_j|(N_1 - V_1)^-, \quad \text{for some } 1 \le j \le m .$$

We claim that even g is admissibly isotopic to g_j. Since g is arbitrarily given, this would prove 3.3.

Define $h = g_j^{-1} g$. Then $h(N\ ) = N$ and $h|(N\ - V\ )^- = \mathrm{id}$. It remains to show that h is admissibly isotopic to the identity. By the following assertion, it suffices to prove that the restriction $h|H$ can be admissibly isotoped in H into the identity.

3.4. *Assertion. Suppose that* $h\ H$ *is admissibly isotopic in* H *to the identity. Then* h *is admissibly isotopic in* $(N_0, \underline{\underline{n}}_0)$ *to the identity.*

Since $(F, \underline{\underline{f}})$ is not an annulus or torus, it is easily seen that there is an admissible isotopy ϕ_t, $t \in I$, of $h|H$ with $\phi_t(H) = H$ and $\phi_t(V_1 \cap H) = V_1 \cap H$, for all $t \in I$, and $\phi_1 = \mathrm{id}|H$ (apply the theorems of Nielsen and Baer).

Removing all the components from V_1 which are regular neighbourhoods of some side of $(N_1, \underline{\underline{n}}_1)$ we obtain an essential F-manifold V_1'. $(N_0, \underline{\underline{n}}_0)$ is a simple 3-manifold. Hence every component of V_1' and every component of $(N_1 - V_1')^-$ has to meet $U(F)$. More precisely, we have a partition of N_0 consisting of the following parts:

1. the regular neighbourhood of F, $U(F)$,
2. components of $(N_1 - V_1')^-$ which are not I-bundles over the square or annulus,

3. I-bundles of V_1' which meet $U(F)$ in lids, but which are not I-bundles over the square or annulus,

4. I-bundles over discs which do not meet $U(F)$ in lids, and Seifert fibre spaces over discs with at most one exceptional fibre (i.e. solid tori).

By 1.4., the parts described in 2. meet H in an essential surface whose components are different from inner squares or annuli.

h is an admissible homeomorphism which preserves this partition, and, of course, ϕ_t can be extended to an admissible isotopy h_t, $t \in I$, of h which preserves the partition and which is constant outside a regular neighbourhood of H. In fact, h_t may be chosen such that, in addition, h_1 is the identity on $U(F)$ and on all parts of the partition described in 2. To see this note first that $U(F)$ is a product I-bundle and that the regular neighbourhood of H intersects every part of the partition in a system of product I-bundles. Then recall that $h|(N_1 - V_1)^-$ is the identity, and observe that every admissible homeomorphism of an I-bundle which is the identity on the lids can be admissibly isotoped into the identity (see proof of 3.5. in [Wa 3]), and this isotopy may be chosen to be constant on the lids provided the base of the I-bundle is not an annulus. Moreover, this isotopy may be chosen to be constant on all the sides of the I-bundle on which the homeomorphism is already the identity. Hence, since every part of the partition meets $U(F)$, this implies that h_t may be chosen so that, in addition, h_1 is the identity on all the parts as described in 3. Therefore we may suppose that h_1 is the identity on all parts except those described in 4.

So, let X be a submanifold of the partition as described in 4. Let A be the union of all the sides of X which are contained in parts of the partition different from X. Then it follows from the properties of X that A is connected, for otherwise we find an essential square or annulus in X which does not meet $U(F)$, which is impossible since $(N_0, \underline{\underline{n}}_0)$ is simple. Hence, by the properties of X, every admissible homeomorphism of X which is the identity on A can be admissibly isotoped to the identity, using an isotopy which is constant on A. By the suppositions on the isotopy h_t, $t \in I$, this implies the assertion.

In order to prove the supposition of 3.4., i.e. that $h|H$ is admissibly isotopic in H to the identity, we introduce the concept of "good submanifolds"

An essential F-manifold W in $(N_1, \underline{\underline{n}}_1)$ is called a *good submanifold,* if

(i) W meets H in an essential surface G with the property: no component of $(H-G)^-$ is an inner square or annulus in H which meets a component of G which is also an inner square or annulus,

(ii) there is an admissible isotopy of h which preserves $U(F)$ and which moves h so that afterwards $h(W) = W$ and $h|(H-G)^- = id|(H-G)^-$.

In the remainder of the proof the property (i) of an essential surface in H will be called the *square- and annulus-property.*

3.5. Assertion. There is at least one good submanifold in $(N_1, \underline{\underline{n}}_1)$.

We obtain a good submanifold by modifying the characteristic submanifold V_1 of $(N_1, \underline{\underline{n}}_1)$. Indeed, by what we have seen so far, V_1 satisfies (ii), and $V_1 \cap H$ is an essential surface in H. Suppose that there is a component A of $(H - V_1)^-$ which is an inner square (resp. annulus) in H and which meets a component B of $V_1 \cap H$ which itself is also an inner square (resp. annulus) in H. Let $U(B)$ be a regular neighbourhood of B in $(N_1, \underline{\underline{n}}_1)$, and define $V_1' = (V_1 - U(B))^-$. Then V_1' satisfies (ii), for V_1 satisfies (ii) and since $h|B \cup A$ is isotopic to the identity, by an isotopy in $B \cup A$ which is constant on $\partial B - A$. Thus, after finitely many steps, we obtain an admissible F-manifold with (i) and (ii). Removing trivial components from this F-manifold, we finally get a good submanifold. This completes the proof of 3.5.

To continue the proof, let W be any good submanifold in $(N_1, \underline{\underline{n}}_1)$. A moment's reflection shows that we may suppose that W is chosen so that, for every good submanifold W' with $W' \subset W$, the essential surface $W \cap H$ can be admissibly isotoped in H into $W' \cap H$.

3.6. Assertion. W' *can be admissibly isotoped in* $(N_1, \underline{\underline{n}}_1)$ *so that afterwards*

$$W \cap H = d(W \cap H) ,$$

where $d : H \to H$ *is the involution given by the reflections in the fibres of the product I-bundle* $U(F)$.

Define $G = W \cap H$, and suppose that G is in a very good position to dG. Of course, this position can always be obtained, using an admissible isotopic deformation of W in $(N_1, \underline{\underline{n}}_1)$. Denote by G' the *essential intersection* of G and dG, i.e. the largest essential surface contained in $G \cap dG$. Then, of course, $(H - G')^-$ is the essential union of $(H - G)^-$ and $(H - dG)^-$.

$U(F)$ is a product I-bundle. Setting $X = U(F)$ and $h = h|U(F)$, we see that we may apply 2.3. Hence it follows the existence of an admissible isotopy ϕ_t, $t \in I$, of $h|H$, with $\phi_t(H) = H$ and $\phi_t(G) = G$, for all $t \in I$, such that $\phi_1(H-G')^- = \mathrm{id}\ (H-G')^-$.

Let G_1 be a component of G. It is easily checked that for 3.6. it suffices to show that G_1 can be admissibly contracted in H to a component of G' contained in G_1 (recall that W has property (i)).

Case 1 G_1 *is an inner square or annulus in* H.

It follows from the existence of the isotopy ϕ_t that G_1 contains at least one component G_1' of the essential intersection G'. For otherwise, removing trivial components from $(W - U(G_1))^-$ (if necessary) we obtain a good submanifold W' such that G cannot be admissibly isotoped into $W \cap H$ (recall that G has the square- and annulus-property), where $U(G_1)$ is a regular neighbourhood of G_1 in $(N_1, \underline{\underline{n}}_1)$. This, however, contradicts our choice of W. Since G_1' is an essential surface, it is an inner square or annulus in H. Thus, of course, G_1 can be admissibly contracted in H to G_1'.

Case 2 G_1 *is not an inner square or annulus in* H.

Recall that G_1 is a component of $H \cap W$. Let X be the component of W which contains G_1. Since we are in Case 2 and since W is an essential F-manifold, it follows that X is an I-bundle and that G_1 is one lid of X.

Let $p : X \to B$ be the projection, and let $G_1^+ = (\partial X - p^{-1}\partial B)^-$. Then G_1 is a component of G_1^+. Denote by $e : G_1^+ \to G_1^+$ the involution given by the reflections in the I-fibres of X. As boundary-pattern of G_1^+, we fix the boundary-pattern induced by $\underline{\underline{n}}_0$, together with the set of components of $(\partial G_1^+ - \partial H)^-$. Then e is an admissible involution of G_1^+.

Define $G_1'' = G' \cap G_1^+$. Since G' is the essential intersection of G and dG, G_1'' is an essential surface in H. Since G is in a very good position to dG, it follows that G_1'' is even an essential surface in G_1^+.

Moreover, we may suppose that W is admissibly isotoped so that G_1'' is in a very good position with respect to $e(G_1'')$.

Since W is a good submanifold, we may suppose that h is admissibly isotoped so that $h(W) = W$ and $h|(H - W)^- = id$. In particular, $h|X$ is an admissible homeomorphism of X. Setting $h = h|X$ and $G = G_1'$, we claim that 2.3. may be applied. For this it remains to show that $h|X$ can be admissibly isotoped in X so that afterwards $h|(G_1^+ - G_1'')^- = id$. But this follows immediately from the existence of the admissible isotopy ϕ_t of $h|H$ defined in the beginning of 3.6.

Now, by .3., $h|X$ can be admissibly isotoped in X so that afterwards $h\, p^{-1}pR = id$, where R is the essential union of $(G_1^+ - G_1'')^-$ and $(G_1^+ - eG_1'')^-$. In general, however, this isotopy cannot be chosen to be constant on $(\partial X - \partial N_1)^-$. Therefore we also fix a regular neighbourhood U of $(\partial X - \partial N_1)^-$ in X, and we define

$$W' = (W-X) \cup p^{-1}pR \cup U .$$

Then it is easily checked that W' is an essential F-manifold in $(N_1, \underline{\underline{n}}_1)$ with property (ii). Without loss of generality, W' also has property (i), i.e. W' is a good submanifold. For, if this is not the case, we simply have to add the components of $(X - W')^-$ to W' which are I-bundles over the square or annulus (recall that W has property (i)).

By our choice of W, the essential surface $H \cap W$ can be admissibly isotoped in H into $W' \cap H$. In particular, $H \cap X$ can be admissibly isotoped into $H \cap p^{-1}pR$. By definition of R, this implies that G_1 can be admissibly contracted to some component of $G_1 \cap G'$.

This completes the proof of 3.6.

Since, by 3.6., we may suppose that $W \cap H = d(W \cap H)$, there is a system Z of I-bundles in $U(F)$ with $Z \cap H = W \cap H$. The submanifold

$$W^+ = W \cup Z$$

consists of essential I-bundles, Seifert fibre spaces, and Stallings manifolds in $(N_0, \underline{\underline{n}}_0)$.

Since $N_0 = M_{2i}$ is a simple 3-manifold, the characteristic submanifold V_0 of $(N_0,\underline{\underline{n}}_0)$ is trivial. Hence also W^+ is trivial, i.e. W^+ is contained in a regular neighbourhood of some sides of $(N_0,\underline{\underline{n}}_0)$ (note that, by the suppositions of 3.3., N_0 is not a Stallings manifold and that, by 1. of 1.2., $(\partial W^+ - \partial N_0)^-$ can be admissibly isotoped into V_0). In particular $H \cap W$ is contained in a regular neighbourhood of some sides of H. Hence it follows from property (ii) of W, that $h|H$ can be admissibly isotoped in H into the identity. This completes the proof of 3.3. q.e.d.

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